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4 Analytic Geometry

1. *Rectangular Coordinates*

The points in a plane may be placed in one-to-one correspondence with pairs of real numbers. A common method is to use perpendicular lines that are horizontal and vertical and intersect at a point called the *origin*. These two lines constitute the coordinate axes; the horizontal line is the x-axis and the vertical line is the y-axis. The positive direction of the x-axis is to the right whereas the positive direction of the y-axis is up. If P is a point in the plane one may draw lines through it that are perpendicular to the x- and y-axes (such as the broken lines of Figure 4.1). The lines intersect the x-axis at a point with coordinate x_1 and the y-axis at a

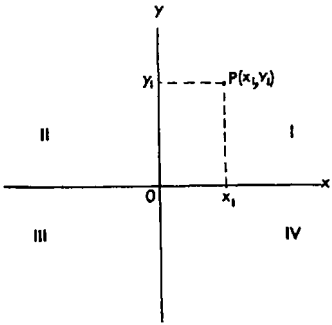

FIGURE 4.1. Rectangular coordinates.

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point with coordinate y_1. We call x_1 the x-coordinate or *abscissa* and y_1 is termed the y-coordinate or *ordinate* of the point P. Thus, point P is associated with the pair of real numbers (x_1, y_1) and is denoted $P(x_1, y_1)$. The coordinate axes divide the plane into quadrants I, II, III, and IV.

2. *Distance between Two Points; Slope*

The distance d between the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In the special case when P_1 and P_2 are both on one of the coordinate axes, for instance, the x-axis,

$$d = \sqrt{(x_2 - x_1)^2} = |x_2 - x_1|,$$

or on the y-axis,

$$d = \sqrt{(y_2 - y_1)^2} = |y_2 - y_1|.$$

The midpoint of the line segment P_1P_2 is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

The slope of the line segment P_1P_2, provided it is not vertical, is denoted by m and is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

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The slope is related to the angle of inclination α (Figure 4.2) by

$$m = \tan \alpha$$

Two lines (or line segments) with slopes m_1 and m_2 are perpendicular if

$$m_1 = -1/m_2$$

and are parallel if $m_1 = m_2$.

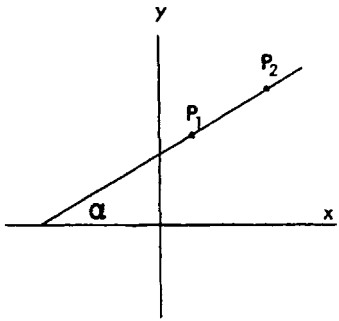

FIGURE 4.2. The angle of inclination is the smallest angle measured counterclockwise from the positive x-axis to the line that contains P_1P_2.

3. *Equations of Straight Lines*

A *vertical* line has an equation of the form

$$x = c$$

where $(c,0)$ is its intersection with the x-axis. A line of slope m through point (x_1, y_1) is given by

$$y - y_1 = m(x - x_1)$$

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Thus, a *horizontal line* (slope = 0) through point (x_1, y_1) is given by

$$y = y_1.$$

A nonvertical line through the two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by either

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

or

$$y - y_2 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_2).$$

A line with x-intercept a and y-intercept b is given by

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad (a \neq 0, b \neq 0).$$

The *general equation* of a line is

$$Ax + By + C = 0$$

The *normal form* of the straight line equation is

$$x \cos \theta + y \sin \theta = p$$

where p is the distance along the normal from the origin and θ is the angle that the normal makes with the x-axis (Figure 4.3).

The general equation of the line $Ax + By + C = 0$ may be written in normal form by dividing by $\pm\sqrt{A^2 + B^2}$, where the plus sign is used when C is negative and the

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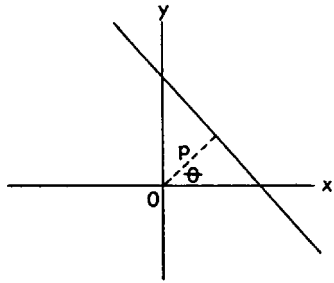

FIGURE 4.3. Construction for normal form of straight line equation.

minus sign is used when C is positive:

$$\frac{Ax+By+C}{\pm\sqrt{A^2+B^2}}=0,$$

so that

$$\cos\theta=\frac{A}{\pm\sqrt{A^2+B^2}}, \qquad \sin\theta=\frac{B}{\pm\sqrt{A^2+B^2}}$$

and

$$p=\frac{|C|}{\sqrt{A^2+B^2}}.$$

4. *Distance from a Point to a Line*

The perpendicular distance from a point $P(x_1, y_1)$ to the line $Ax+By+C=0$ is given by d

$$d=\frac{Ax_1+By_1+C}{\pm\sqrt{A^2+B^2}}.$$

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5. *Circle*

The general equation of a circle of radius r and center at $P(x_1, y_1)$ is

$$(x-x_1)^2+(y-y_1)^2=r^2.$$

6. *Parabola*

A parabola is the set of all points (x, y) in the plane that are equidistant from a given line called the *directrix* and a given point called the *focus*. The parabola is symmetric about a line that contains the focus and is perpendicular to the directrix. The line of symmetry intersects the parabola at its *vertex* (Figure 4.4). The eccentricity $e=1$.

The distance between the focus and the vertex, or vertex and directrix, is denoted by $p(>0)$ and leads to one of the following equations of a parabola with vertex at the origin (Figures 4.5 and 4.6):

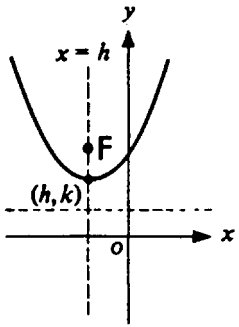

FIGURE 4.4. Parabola with vertex at (h, k). F identifies the focus.

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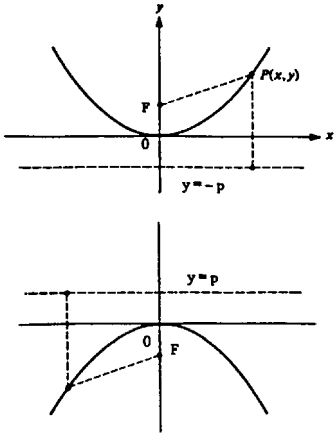

FIGURE 4.5. Parabolas with y-axis as the axis of symmetry and vertex at the origin. (Upper) $y = \dfrac{x^2}{4p}$; (lower) $y = -\dfrac{x^2}{4p}$.

$$y = \frac{x^2}{4p} \qquad \text{(opens upward)}$$

$$y = -\frac{x^2}{4p} \qquad \text{(opens downward)}$$

$$x = \frac{y^2}{4p} \qquad \text{(opens to right)}$$

$$x = -\frac{y^2}{4p} \qquad \text{(opens to left)}$$

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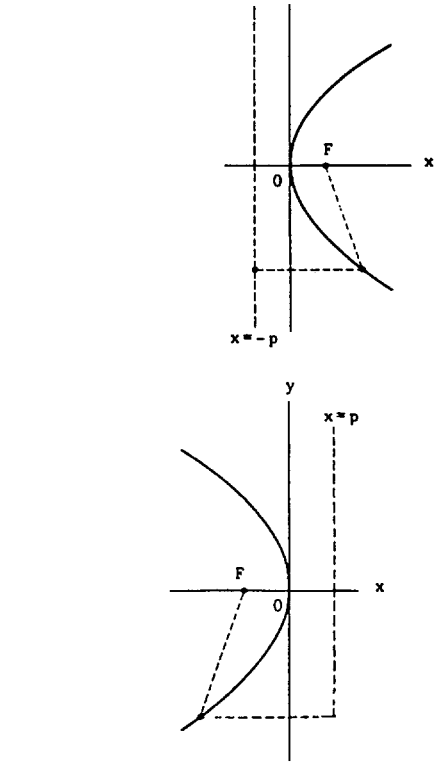

FIGURE 4.6. Parabolas with x-axis as the axis of symmetry and vertex at the origin. (Upper) $x = \dfrac{y^2}{4p}$; (lower) $x = -\dfrac{y^2}{4p}$.

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For each of the four orientations shown in Figures 4.5 and 4.6, the coresponding parabola with vertex (h,k) is obtained by replacing x by $x-h$ and y by $y-k$. Thus, the parabola in Figure 4.7 has the equation

$$x-h=-\frac{(y-k)^2}{4p}.$$

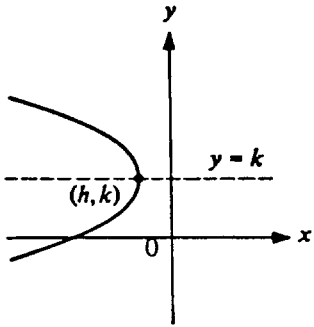

FIGURE 4.7. Parabola with vertex at (h,k) and axis parallel to the x-axis.

7. *Ellipse*

An ellipse is the set of all points in the plane such that the sum of their distances from two fixed points, called *foci*, is a given constant $2a$. The distance between the foci is denoted $2c$; the length of the major axis is $2a$, whereas the length of the minor axis is $2b$ (Figure 4.8) and

$$a=\sqrt{b^2+c^2}\,.$$

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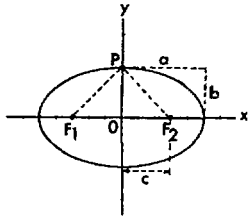

FIGURE 4.8. Ellipse; since point P is equidistant from foci F_1 and F_2 the segments F_1P and $F_2P = a$; hence $a = \sqrt{b^2 + c^2}$.

The eccentricity of an ellipse, e, is < 1. An ellipse with center at point (h, k) and major axis *parallel to the x-axis* (Figure 4.9) is given by the equation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

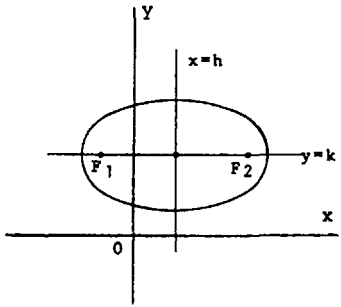

FIGURE 4.9. Ellipse with major axis parallel to the x-axis. F_1 and F_2 are the foci, each a distance c from center (h, k).

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An ellipse with center at (h,k) and major axis parallel to the y-axis is given by the equation (Figure 4.10)

$$\frac{(y-k)^2}{a^2}+\frac{(x-h)^2}{b^2}=1.$$

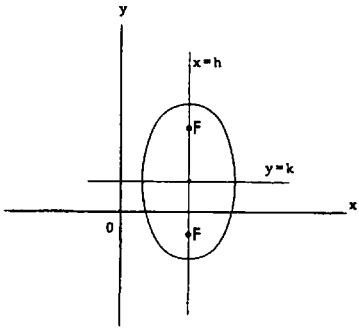

FIGURE 4.10. Ellipse with major axis parallel to the y-axis. Each focus is a distance c from center (h,k).

8. *Hyperbola ($e > 1$)*

A hyperbola is the set of all points in the plane such that the difference of its distances from two fixed points (foci) is a given positive constant denoted $2a$. The distance between the two foci is $2c$ and that between the two vertices is $2a$. The quantity b is defined by the equation

$$b=\sqrt{c^2-a^2}$$

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and is illustrated in Figure 4.11, which shows the construction of a hyperbola given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

When the focal axis is parallel to the y-axis the equation of the hyperbola with center (h,k) (Figures 4.12 and 4.13) is

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1.$$

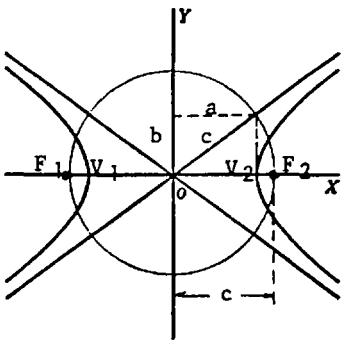

FIGURE 4.11. Hyperbola; V_1, V_2 = vertices; F_1, F_2 = foci. A circle at center 0 with radius c contains the vertices and illustrates the relation among a, b, and c. Asymptotes have slopes b/a and $-b/a$ for the orientation shown.

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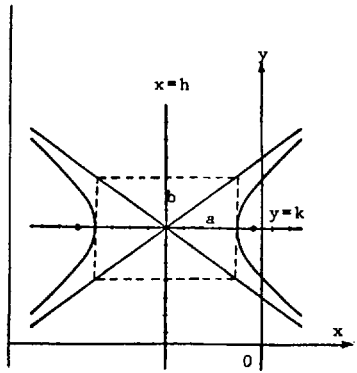

FIGURE 4.12. Hyperbola with center at (h, k): $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$; slopes of asymptotes $\pm b/a$.

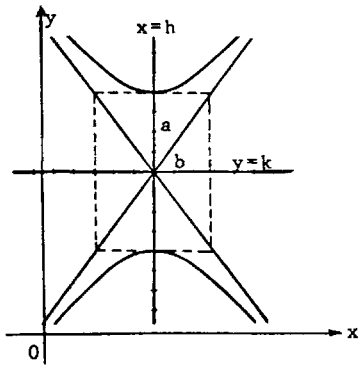

FIGURE 4.13. Hyperbola with center at (h, k): $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$; slopes of asymptotes $\pm a/b$.

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If the focal axis is parallel to the x-axis and center (h,k), then

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

9. *Change of Axes*

A change in the position of the coordinate axes will generally change the coordinates of the points in the plane. The equation of a particular curve will also generally change.

- *Translation*

When the new axes remain parallel to the original, the transformation is called a *translation* (Figure 4.14). The new axes, denoted x' and y', have origin $0'$ at (h,k) with reference to the x and y axes.

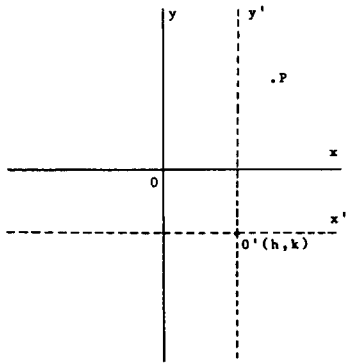

FIGURE 4.14. Translation of axes.

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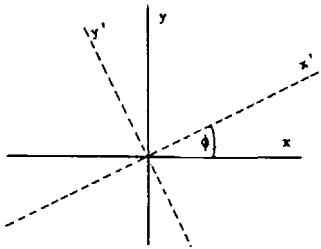

FIGURE 4.15. Rotation of axes.

A point P with coordinates (x, y) with respect to the original has coordinates (x', y') with respect to the new axes. These are related by

$$x = x' + h$$

$$y = y' + k$$

For example, the ellipse of Figure 4.10 has the following simpler equation with respect to axes x' and y' with the center at (h, k):

$$\frac{y'^2}{a^2} + \frac{x'^2}{b^2} = 1.$$

- *Rotation*

When the new axes are drawn through the same origin, remaining mutually perpendicular, but tilted with respect to the original, the transformation is one of rotation. For angle of rotation ϕ (Figure 4.15), the coordinates (x, y) and (x', y') of a point P are related by

$$x = x' \cos \phi - y' \sin \phi$$

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$$y = x' \sin \phi + y' \cos \phi$$

10. General Equation of Degree Two

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

Every equation of the above form defines a conic section or one of the limiting forms of a conic. By rotating the axes through a particular angle ϕ, the xy-term vanishes, yielding

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0$$

with respect to the axes x' and y'. The required angle ϕ (see Figure 4.15) is calculated from

$$\tan 2\phi = \frac{B}{A - C}, \qquad (\phi < 90°).$$

11. Polar Coordinates (Figure 4.16)

The fixed point O is the origin or *pole* and a line OA drawn through it is the polar axis. A point P in the plane is determined from its distance r, measured from

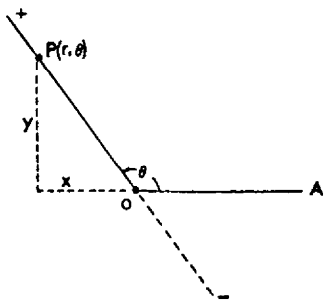

FIGURE 4.16. Polar coordinates.

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O, and the angle θ between OP and OA. Distances measured on the terminal line of θ from the pole are positive, whereas those measured in the opposite direction are negative.

Rectangular coordinates (x, y) and polar coordinates (r, θ) are related according to

$$x = r\cos\theta, \qquad y = r\sin\theta$$

$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x.$$

Several well-known polar curves are shown in Figures 4.17 to 4.21.

The polar equation of a conic section with focus at the pole and distance $2p$ from directrix to focus is either

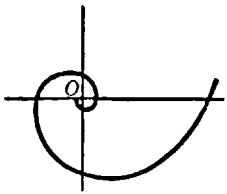

FIGURE 4.17. Polar curve $r = e^{a\theta}$.

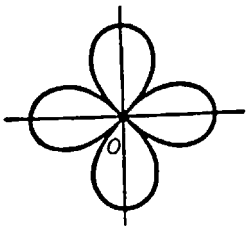

FIGURE 4.18. Polar curve $r = a\cos 2\theta$.

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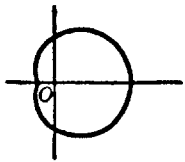

FIGURE 4.19. Polar curve $r = 2a\cos\theta + b$.

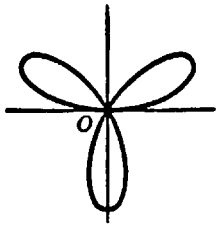

FIGURE 4.20. Polar curve $r = a\sin 3\theta$.

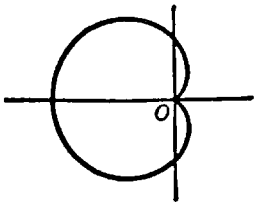

FIGURE 4.21. Polar curve $r = a(1 - \cos\theta)$.

$$r = \frac{2ep}{1 - e\cos\theta} \qquad \text{(directrix to left of pole)}$$

or

$$r = \frac{2ep}{1 + e\cos\theta} \qquad \text{(directrix to right of pole)}$$

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The corresponding equations for the directrix below or above the pole are as above, except that sin θ appears instead of cos θ.

12. *Curves and Equations*

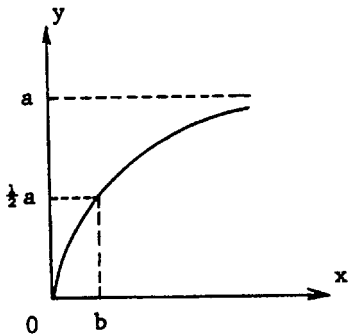

FIGURE 4.22. $y = \dfrac{ax}{x+b}$

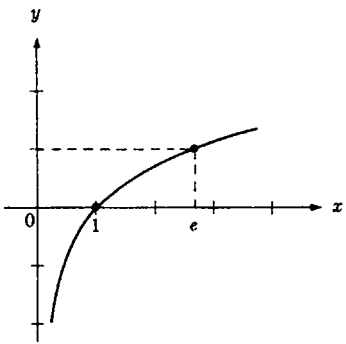

FIGURE 4.23. $y = \log x$.

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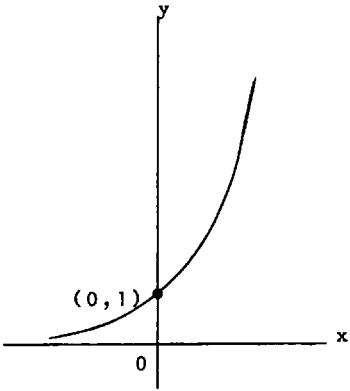

FIGURE 4.24. $y = e^x$.

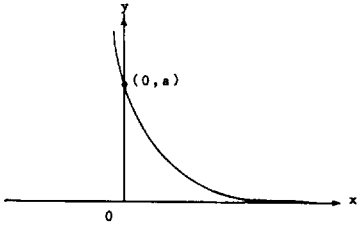

FIGURE 4.25. $y = ae^{-x}$.

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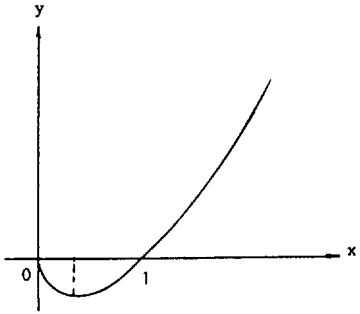

FIGURE 4.26. $y = x \log x$.

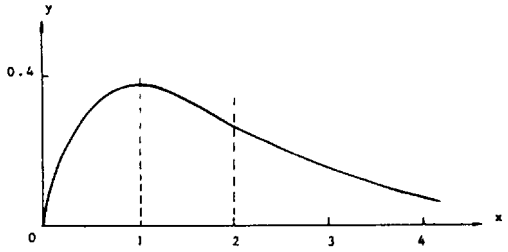

FIGURE 4.27. $y = xe^{-x}$.

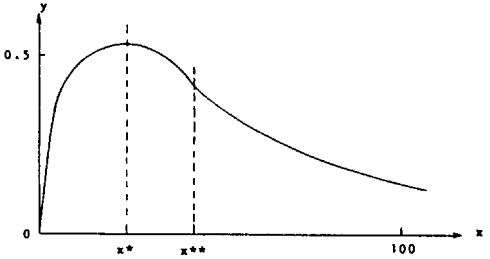

FIGURE 4.28. $y = e^{-ax} - e^{-bx}$, $0 < a < b$ (drawn for $a = 0.02$, $b = 0.1$, and showing maximum and inflection).

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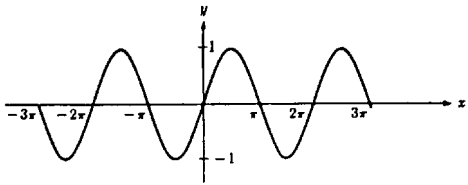

FIGURE 4.29. $y = \sin x$.

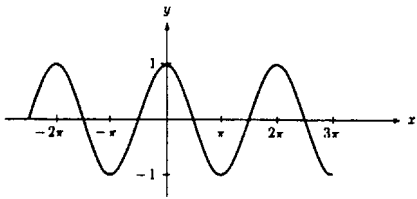

FIGURE 4.30. $y = \cos x$.

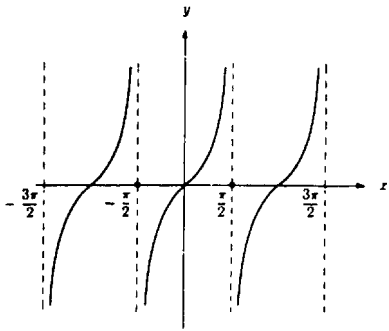

FIGURE 4.31. $y = \tan x$.

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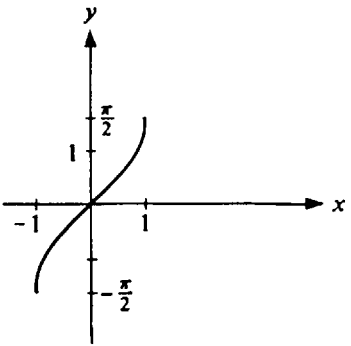

FIGURE 4.32. $y = \arcsin x$.

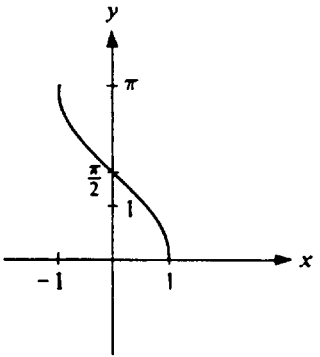

FIGURE 4.33. $y = \arccos x$.

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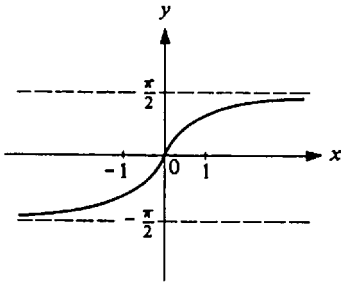

FIGURE 4.34. $y = \arctan x$.

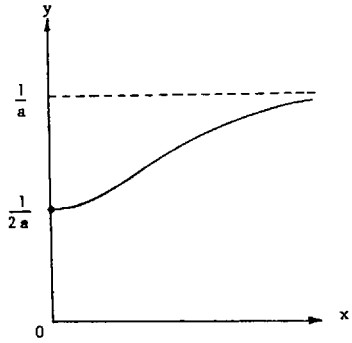

FIGURE 4.35. $y = e^{bx}/a(1 + e^{bx})$, $x \geq 0$ (logistic equation).

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